Lecture – 40.

Gauss's Divergence Theorem

Imp. Results: (From MCQ point of View)

If S is the boundary of a closed and bounded region D and S is a orientable surface, then:

1. $\iint_S (\vec{r}\cdot\hat{n})\, dV = 3V$; $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ and V is volume of region.

2. $\iint_S (\vec{a}\cdot\hat{n})\, dA = 0$; where $\vec{a}$ is a constant vector $\vec{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

3. $\iint_S (\text{curl}\,\vec{V}\cdot\hat{n})\, dA = 0$

4. $\iint_S r^n(\vec{r}\cdot\hat{n})\, dA = (n+3)\iiint_D r^n\, dv$; $n \neq -3$
   $r^2 = x^2 + y^2 + z^2$

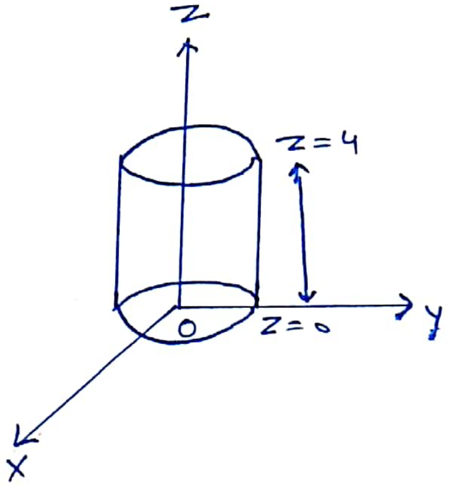
5. $\iint_S (\nabla r^2)\cdot\hat{n}\, dA = 6V$; $r^2 = x^2 + y^2 + z^2$ and V is volume

Q: Evaluate $\iint_S (\vec{V}\cdot\hat{n})\,dA$, using Gauss's div. thm.

Where $\vec{V} = (3x^2\hat{i} + 6y^2\hat{j} + z\hat{k})$ and D is the region bounded by the closed cylinder $x^2+y^2=16$, $z=0$ and $z=4$.

Sol: $\vec{V} = (3x^2\hat{i} + 6y^2\hat{j} + z\hat{k})$

$\Rightarrow$ $\operatorname{div}(\vec{V}) = (6x + 12y + 1)$



By Gauss's div. thm:

$$\iint_S (\vec{V}\cdot\hat{n})\,dA = \iiint_D (\operatorname{div}\vec{V})\,dV$$

$$= \iiint_D (6x+12y+1)\,dV$$

$$= \int_{z=0}^{4}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (6x+12y+1)\,dy\,dx\,dz$$

Since x and y are odd functions

$$\Rightarrow \iint_S (\vec{V}\cdot\hat{n})\,dA = (4)\cdot(2)\,(2)\int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}} dy\,dx$$

$$= 16\int_{x=0}^{4}\sqrt{16-x^2}\,dx$$

$$= \left[16\ \tfrac{1}{2}x\sqrt{16-x^2} + \tfrac{16}{2}\sin^{-1}\left(\tfrac{x}{4}\right)\right]_0^4$$

$$= 64\pi$$ Ans.

Evaluate $\iint_S (\overline{V}\cdot\hat{n})\,dA$, Using Gauss's div. thm. (3).

Where $\overline{V} = (x^2 z)\hat{i} + y\hat{j} - (xz^2)\hat{k}$ and S is the boundary of the region bounded by paraboloid $z = x^2+y^2$ and the plane $z = 4y$.

Sol: $\overline{V} = x^2 z\hat{i} + y\hat{j} - xz^2\hat{k}$

$\Rightarrow \operatorname{div}\overline{V} = (\cancel{2xz} + 1 - \cancel{2xz}) = 1.$

By Gauss's div. thm.

$$\iint_S (\overline{V}\cdot\hat{n})\,dA = \iiint_D (\operatorname{div}\overline{V}).dv = \iiint_D dv$$

$$= \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}}\int_{z=x^2+y^2}^{4y} dz\,dx\,dy$$

$[\,x^2+y^2 = 4y \Rightarrow x = \pm\sqrt{4y-y^2}\,]$

$$= \int_{y=0}^{4}\int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y - x^2 - y^2)\,dx\,dy$$

$$= \int_{y=0}^{4} \frac{4}{3}(4y-y^2)^{3/2}\,dy$$

$$= \frac{4}{3}\int_{y=0}^{4} [4 - (y-2)^2]^{3/2}\,dy$$

Let $(y-2) = 2\sin t \Rightarrow dy = 2\cos t\,dt$

$$\Rightarrow \iint_S (\overline{V}\cdot\hat{n})\,dA = \frac{4}{3}\int_{-\pi/2}^{\pi/2} 16.\cos^4 t\,dt = \frac{4}{3}(32)\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$= 8\pi \quad \text{Ans.}$$

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